Skyrmions in quantum Hall ferromagnets as spin-waves bound to unbalanced magnetic flux quanta

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Abstract

A microscopic description of (baby) skyrmions in quantum Hall ferromagnets is derived from a scattering theory of collective (neutral) spin modes by a bare quasiparticle. We start by mapping the low lying spectrum of spin-waves in the uniform ferromagnet onto that of free moving spin excitons, and then we study their scattering by the defect of charge. In the presence of this disturbance, the local spin stiffness varies in space, and we translate it into an inhomogeneous metric in the the Hilbert space supporting the excitons. An attractive potential is then required to preserve the symmetry under global spin rotations, and it traps the excitons around the charged defect. The quasiparticle now carries a spin texture. Textures containing more than one exciton are described within a mean-field theory, the interaction among the excitons being taken into account through a new renormalization of the metric. The number of excitons actually bound depends on the Zeeman coupling, that plays the same role as a chemical potential. For small Zeeman energies, the defect binds many excitons which condensate. As the bound excitons have a unit of angular momentum, provided by the quantum of magnetic flux left unbalanced by the defect of charge, the resulting texture turn out to be a topological excitation of charge 1. Its energy is that given by the non-linear sigma model for the ground state in this topological sector, i.e. the texture is a skyrmion.

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I. INTRODUCTION

Low energy excitations of two dimensional systems in the quantum Hall regime reveal a rich variety of new physical phenomena. Specially striking are the cases of filling factors which imply a ground state which is a quantum Hall ferromagnet (QHF) in the lowest Landau level (LLL), because their charged quasiparticles may carry non-trivial spin textures. In the limit of low Zeeman coupling, these textures have been identified as skyrmions with real and topological charge equal to one, being successfully described by means of a nonlinear $\sigma-$model (NLσM).[1] For realistic Zeeman energies, the quasiparticles may be regarded as distorted (baby) skyrmions with the same topological winding number but a finite size determined by the competition between the Zeeman coupling and the electronic interaction. Classical field theories fail to give an accurate description of these localized spin textures for which quantum fluctuations are important. Various microscopic approaches have been used to attack this task: (i) mean field descriptions of the Hartree-Fock type[2] (ii) different kinds of variational schemes[3,4] and (iii) microscopic wave-functions, with well defined quantum numbers, obtained from small systems.[5]

In this paper we face the problem of baby-skyrmions in actual QHF from a completely different point of view. We present a description of these textures derived from a scattering theory of collective (neutral) spin modes by a bare charged defect. By bare charged defect we mean the quasiparticle in the case of Zeeman coupling much larger than the electronic interaction. Our scheme starts with long wavelength spin-waves in the uniform QHF, where they have been shown to behave as free moving electron-hole eh pairs (excitons) of vanishing electric dipolar moment. These excitons exhaust the low lying spectrum of neutral excitations of the QHF. Our aim is to show that these modes may destabilize the bare quasiparticle by becoming bound to it and raise a charged spin texture. We have developed an effective Hamiltonian to describe this scattering event. The actual number of bound excitons, *i.e.* of spin flips, in the lowest energy texture is a matter of the Zeeman coupling, that in our picture plays the role of a chemical potential. To consider skyrmions as spin-waves bound to defects has been sometimes suggested but never developed.[4,6]

Before going into details, let us summarize the main ideas of our work. As mentioned above, the low lying spectrum of a uniform QHF, in the strong magnetic field limit, can be mapped to that of a gas of free eh pair of vanishing electrical dipolar moment and well defined linear momentum.[7] We assume that this mapping is still possible in the presence of the bare charged defect, although the excitons are not free any more. They interact with the disturbance as well as among themselves. The effect of the disturbance is threefold:

i) As the quasiparticle is identical to one of the components of the exciton (the spin up hole or the spin down electron), Pauli principle does not allow them to share spatial position. In other words, the exciton moves in a non-uniform space because its position is less probable near the defect than far from it. We will show that this leads to an inhomogeneous metric in the Hilbert space supporting the wave function of the exciton.

ii) The extra charge of $\mp\nu$ electrons leaves unbalanced exactly one quantum of magnetic flux. It introduces an Aharonov-Bohm phase that gives a unit(±1) of angular momentum to the excitons in their ground state. Mapped back to the language of spin textures it means vorticity with winding number equal to ±1.

iii) Finally, the sharp localization of the spin of the bare quasiparticle, facing that of

the particles in the QHF (either electrons or holes), is a waste of energy, since a smoother alignment of spins would allow to gain exchange energy. It produces an attractive short-range potential that binds the excitons.

Using simple arguments, we derive in this work explicit expressions for both the new metric and the potential. This kind of treatment, is enough to find the lowest lying bound state of a unique exciton. Comparison of this state with the one obtained by numerical calculations firmly supports our framework.

The next step required for a complete description of charged textures is to let the defect bind more than one exciton, what implies to take into account the interaction among them. For that purpose we use a mean field approximation for the excitons. This approach should be very accurate for textures containing a large number of excitons, in which case we should recover a description for skyrmions of quasi infinite size. We will check that effectively this is our result. As well, it follows from our analysis that it also works quite well even for textures with very few excitons.

The paper is organized as follows: In section II we describe the scattering of one exciton by a charged defect. Since the method does not require a microscopic wave function of the QHF, it is directly applicable to all the cases with filling factor $\nu = 1/(2p+1)$ with p integer. For the sake of simplicity the main body of the paper is devoted to $\nu = 1$, but a discussion of the case $\nu = 1/3$ is also presented at the end of that section II. In order to study a spin texture with a larger size, section III presents our mean field approach for the binding of an increasing number of excitons. A brief summary is presented in section IV.

II. ONE SPIN-WAVE BOUND TO A CHARGED DEFECT

Let us start by describing a spin-wave in a uniform QHF in terms of a single exciton within the formalism later required to study its scattering by a charged defect. We consider the magnetic field high enough to make the LLL approximation. Since the system is invariant under spin rotations around a unitary vector $\mathbf{u}_B$ in the direction of the magnetic field, the Zeeman coupling $\tilde{g}$ gives only an energy shift. Therefore, $\tilde{g}$ is for the moment taken as zero and included a posteriori. As the kinetic energy is quenched in the LLL approximation, the only contribution to the Hamiltonian comes from the interaction between electrons.

A. Single exciton in a uniform ferromagnet

Since all the polarized excitations of the QHF have a finite gap, its lowest lying spectrum is exhausted by the collective spin modes associated to the spontaneous breaking of the rotation symmetry in spin space (spin-waves). These long wavelength textures are made up of non interacting single spin excitons in which one electron flips its spin to down leaving behind a hole in the filled spin up level. The most general form of this kind of excitations is

$$O_\phi^\dagger |F\rangle = \int d\mathbf{r} d\mathbf{r}' \phi(\mathbf{r}, \mathbf{r}') \Psi_\downarrow^\dagger(\mathbf{r}') \Psi_\uparrow(\mathbf{r}) |F\rangle \tag{1}$$

where $|F\rangle$ is the fully polarized ground state and $\Psi_\downarrow^\dagger$ and $\Psi_\uparrow$ are creation operators of the electron and the hole, respectively, projected onto the LLL. The operator $O_\phi^\dagger$ creates a

eh pair in the positions $\mathbf{r}, \mathbf{r}'$ with a probability given by $|\phi(\mathbf{r}, \mathbf{r}')|^2$. The function $\phi(\mathbf{r}, \mathbf{r}')$ characterizes completely the operator $O_\phi^\dagger$, and can be interpreted as the wave function of the exciton. As the *eh* pair is neutral and $|F\rangle$ is translationally invariant, $\phi(\mathbf{r}, \mathbf{r}')$ may be classified by a conserved wave vector $\mathbf{k}$ that plays the role of the total linear momentum of the exciton. Within the LLL approximation, the fully interacting Hamiltonian can be exactly diagonalized to yield:[7]

$$\phi_\mathbf{k}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} e^{i\mathbf{k}\mathbf{R}} e^{iX\Delta y/l_B^2} \mathcal{G}(\Delta \mathbf{r} - l_B^2 \mathbf{k} \times \mathbf{u}_B) \tag{2}$$

$\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$ is the center of mass coordinate with components (X, Y), $\Delta \mathbf{r} = \mathbf{r} - \mathbf{r}'$ is the relative coordinate with components $(\Delta x, \Delta y)$ and l_B is the magnetic length. The function multiplying the plane wave is just the representation of the delta function in the LLL approximation. It is, except for a gauge dependent factor, a gaussian whose width is the magnetic length. This widening expresses that in the LLL the coordinates x and y are conjugated operators, and hence liable to uncertainty. What Eq. (2) tells us is that the electron and the hole move parallel to one another with a constant velocity perpendicular to their electric dipolar moment, given by $\Delta \mathbf{r} = l_B^2 \mathbf{k} \times \mathbf{u}_B$.

Our interest is on the long wavelength limit $k << l_B^{-1}$. In this case the distance between the two opposite charges vanishes, and the corresponding eigenstates read simply

$$\lim_{k l_B \to 0} \phi_\mathbf{k}(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} e^{i\mathbf{k}\mathbf{R}} \delta(\Delta \mathbf{r}). \tag{3}$$

Now, Eq. (1) becomes:

$$O_\phi^\dagger |F\rangle = \int d\mathbf{r} \phi(\mathbf{r}) \Psi_\downarrow^\dagger(\mathbf{r}) \Psi_\uparrow(\mathbf{r}) |F\rangle \tag{4}$$

with a wave function for the exciton simply given by

$$\phi(\mathbf{r}) = \frac{e^{i\mathbf{k}\mathbf{r}}}{\sqrt{2\pi}}. \tag{5}$$

It is worth commenting that the normalization is a result of the fact that

$$\langle F| O_\phi O_\phi^\dagger |F\rangle = 1 \implies \int d\mathbf{r} |\phi(\mathbf{r})|^2 = 1. \tag{6}$$

The energy of these long wavelength excitons results to be quadratic on the linear momentum (wave vector), $\varepsilon_1(\mathbf{k}) = 4\pi \rho_s k^2$, where ρ_s is the spin stiffness of the QHF.[7] Since any excitation $O_\phi^\dagger |F\rangle$ is totally characterized by a function $\phi(\mathbf{r})$ and energy ε_1, it follows[8] that the many-body problem we are considering, can be mapped onto the single particle Hamiltonian that governs the dynamics of a neutral free particle with an effective mass $m^* = (8\pi \rho_s)^{-1}$

$$\mathcal{H} O_\phi^\dagger |F\rangle = \varepsilon O_\phi^\dagger |F\rangle \iff -4\pi \rho_s \nabla^2 \phi = \varepsilon \phi \tag{7}$$

$\mathcal{H}$ being the fully interacting many-body Hamiltonian.

4

B. Scattering of a single exciton by a defect of charge ± 1

Our aim is to describe (baby)skyrmions as bound states of spin-waves to a charged defect. Hereafter, we take a hole in the ferromagnetic ground state as the bare defect. So, we will obtain an antiskyrmion, while a skyrmion should be produced by taking as a bare defect an electron with opposite spin to that of the electrons in the ferromagnetic ground state. Within the framework presented in the previous subsection, the problem reduces to study the dynamics of the otherwise free moving exciton $\phi_1(\mathbf{r})$ in the presence of the bare quasiparticle. Physically we expect the defect to influence the exciton dynamics in two different ways:

First, as the quasihole (quasielectron) is identical to one of the components of the exciton (the spin up hole or the spin down electron respectively), Pauli principle forbids them to come together. In other words, the exciton moves in a non-uniform space because its position is less probable near the defect than far from it. As well, the extra charge unbalances the commensurability between the number of particles and quanta of magnetic flux. The unpaired quantum of flux introduces an Aharonov-Bohm phase.

Second, the spin up quasihole (spin down quasielectron) has no exchange interaction with spin down holes (spin up electrons) in the filled (either with holes or electrons) level. In order to lower the energy, it is preferable to have a smoother spin field which allows to gain exchange energy. Since the exciton is described by a spin-flip operator, it feels the exchange field as an attractive effective potential in the region of the defect.

Let us follow these physical ideas to derive an effective Hamiltonian describing the dynamics of a single exciton, $\phi_1(\mathbf{r})$, in the presence of a bare quasiparticle. To preserve the symmetry of the problem we use symmetric gauge centered in the position of the defect. Single particle wave functions are

$$\varphi_m(z) = \frac{1}{\sqrt{2^{m+1}\pi m!}} z^m e^{-|z|^2/4l_B^2} \tag{8}$$

where $z = x + iy$, so that hereafter we replace $\mathbf{r}$ by z. The extra hole occupies the single particle state with $m = 0$ and we denote the state containing the bare quasiparticle by $|^0F\rangle$. What is now the Hilbert space of functions $\phi_1(z)$ for the exciton? As a consequence of the fact that $|^0F\rangle$ is not translationally invariant, the normalization condition is not given by Eq. (6) any more. Some straightforward algebra allows to obtain that

$$\langle^0F|O_\phi O_\phi^\dagger|^0F\rangle = 1 \implies \int dz |\phi_1(z)|^2 |\mu_1(z)|^2 = 1 \tag{9}$$

where the new metric for the Hilbert space is given by

$$|\mu_1(z)|^2 = 1 - 2\pi l_B^2 |\varphi_0(z)|^2 \tag{10}$$

In other words, there is change in the scalar product due to the presence of the quasihole occupying $\varphi_0(z)$. This can be understood with the following physical arguments. For an exciton in a uniform QHF, the probability amplitude of being at a position z is, of course, given by $|\phi_1(z)|^2$. In the presence of a flux quantum, the *eh* pair can not move into the region around the defect. In this case, the probability of finding the exciton in a position

z is $|\phi_1(z)|^2$ multiplied by the probability $1 - 2\pi l_B^2 |\varphi_0(z)|^2$ of not having the defect at the same position. This is precisely the new metric $|\mu_1(z)|^2$ we have derived.

The function $|\mu_1(z)|^2$ tends to one for large distances so that the usual metric is recovered far from the defect. On the contrary, $|\mu_1(z)|^2$ goes to zero as r^2 near the origin, where the defect is located. As a crucial result of this behavior around the origin, the new Hilbert space $L^2(\mu_1)$ of exciton wave functions includes new functions that were not square integrable with the usual homogeneous metric (6). In particular, the Hilbert space is expanded with an exciton function that diverging at the origin as r^{-1} has now a finite norm. This new open possibility produces interesting physical results.

The complex function $\mu_1(z)$, whose square modulus gives the new metric, has been obtained hitherto up to a phase factor $e^{if(z)}$. To be consistent with the fact that the hole is linked to an unbalanced quantum of magnetic flux, we chose this factor to be the Aharonov-Bohm phase $e^{i\theta}$, so that

$$\mu_1(z) = e^{i\theta} \sqrt{1 - 2\pi l_B^2 |\varphi_0(z)|^2}. \tag{11}$$

An obvious implication of the new metric is to alter the effective Hamiltonian describing the exciton dynamics. As this Hamiltonian must be hermitian with respect to the new metric, the eigenvalue equation must take the form:

$$H\mu_1 \phi_1 = \varepsilon_1 \mu_1 \phi_1 \tag{12}$$

where H is Hermitian with respect to the usual homogeneous metric. With this, Eq. (7) would read $-4\pi\rho_s \nabla^2 \mu_1 \phi_1 = \varepsilon_1 \mu_1 \phi_1$ and the problem would be formally identical to that of the uniform QHF. However, we have overlooked an important contribution. In its present form, the Hamiltonian does not preserve the symmetry of being invariant under spin rotations in the spin space. We should remember that in our problem the Zeeman and the electronic interaction are separable, and by now we are only taking into account the latest one. Making a rigid rotation of the spin must then cost zero energy. Such a rigid rotation is generated by an operator of the form (4) with $\phi_1(z)$ replaced by a constant (i.e. the spin lowering operator). Hence, $\phi_1(z) \equiv constant$ must be a zero energy eigenstate of the single particle effective Hamiltonian. This condition was guaranteed in the homogeneous metric case, as ∇^2 gives zero when acting on a constant. However, it is no longer true in our new inhomogeneous space, and the effective Hamiltonian needs a position dependent potential correcting the contribution coming from ∇^2

$$V(z) = 4\pi\rho_s \frac{\nabla^2 \mu_1(z)}{\mu_1(z)}. \tag{13}$$

This is an attractive central potential actually capable of binding the excitons. It increases monotonously from the center and it is concentrated in a few magnetic lengths. This is precisely the potential that on physical grounds we expected to describe the overcost of exchange energy due to having a bare quasiparticle.

Therefore, the single particle eigenvalue equation takes the form:

$$4\pi\rho_s \left[-\nabla^2 + \frac{\nabla^2 \mu_1(z)}{\mu_1(z)} \right] \mu_1 \phi_1 = \varepsilon_1 \mu_1 \phi_1. \tag{14}$$

This equation for the exciton in the presence of the defect is the first important result of this paper. For the renormalized function $\mu_1\phi_1$, Eq. (14) is the usual one for a particle in a potential given by Eq. (13). The exact matching between one extra quantum of flux and one lacking charge in the function $\mu_1(z)$ leads to a finite value of (13) at the origin. As well, this value turns out to be $V(0) = -4\pi\rho_s$, which is precisely the energy (measured with respect to $|^0F\rangle$) of the classical infinite-sized skyrmion predicted by the NLσM. In our description, this energy sets a lower bound to the energy of the single exciton described by Eq. (14) as it should be demanded in the sake of consistency.

It is worth commenting that the details of the electron-electron interaction are contained in ρ_s. For instance, for a contact interaction between the electrons, the spin stiffness is zero and there is neither localizing potential nor dispersion relation. (13) Therefore, no excitons are bound to the defect for a contact interaction between the electrons. This is in agreement with the exact result stating that for a contact interaction between the electrons, the spin textures have zero energy.[9]

The shape of the potential (13) is shown in Fig. 1. It is pretty close to $-1/cosh^2(r)$ so that one could approximate the lowest lying solution of Eq. (14) by polynomial expansions, but the consecutive trapping of more than one exciton, that we will discuss latter, requires numerical procedures, so that it is better to start already solving Eq. (14) also by numerical methods. In any case, without any numerical calculation, it is possible to draw the main properties of the lowest lying eigenstate of (14):

- *Energy*: It is a bound state with negative energy

- *Angular momentum*: The product of functions $\mu_1\phi_1$ does not have any angular dependence. It implies that the exciton wave function takes the opposite dependence on θ to that of μ_1, i.e.

$$\phi_1(z) = f(|z|)e^{-i\theta}. \tag{15}$$

  The phase factor $e^{-i\theta}$ gives a winding number -1 to the spin texture $O_\phi^\dagger|^0F\rangle$.

- *Behavior at the origin*: near the origin, the product $\mu_1\phi_1$ tends to a finite constant value. As the metric vanishes as $|z|$ in this region, the limit behavior of the exciton wave function at the origin is

$$\lim_{|z|<l_B} \phi_1(z) = \frac{e^{-i\theta}}{|z|}. \tag{16}$$

  We must stress that in spite of the divergence at the origin, ϕ_1 has finite norm with the new metric μ_1.

- *Behavior at infinity*: For distances much larger than l_B, the solution $\mu_1\phi_1$ decays exponentially to zero. As μ_1 tends to 1, the exciton wave function $\phi_1(z)$ also decays exponentially.

All the above discussed characteristics bring to a very important conclusion:*the bare quasiparticle is able to bind a spin exciton, and raise in this way a charged spin texture.*

7

In its ground state the exciton has a unit of angular momentum that tries to cancel the quantum of flux left unbalanced by the lack of charge. As we will discuss later, for small Zeeman energies, the defect can bind many of such excitons which condensate. Order in the magnetization is developed, the angular dependence of the in-plane component of the order parameter being equal to the angular momentum of the excitons. In other words, a spin texture with unit topological charge appears.

In order to check the quality of our description we can start by discussing the behavior around the origin. The radial dependence $1/|z|$ here obtained, is precisely the one coming out in exact diagonalizations for small quantum dots[5] where the finite size of the droplet cuts off the exponential decay of the exciton wave function, but leaves its core perfectly preserved. Even more important, this slow decay describes as well the long distance behavior of those textures made up of many excitons.[5,9] The exponential decay keeps the exciton localized[3,4], and the size of the texture finite. We will see in the next section that, as more excitons are bound, their mutual repulsion makes them to spread, and the exponential decay of their wave function becomes smoother. Then, for textures made up of many excitons, the wave function of each of them is simply $1/|z|$.

The energy of the texture created by one exciton is a very interesting magnitude because it determines the critical g-factor g_{cr} for the existence of baby skyrmions. For $g > g_{cr}$, the positive Zeeman energy is so high that the exciton becomes unbound and no spin texture can be formed. For this binding energy we obtain $-0.311 \times 4\pi\rho_s$, that would imply a $g_{cr} = 7.9/\sqrt{B}$ T$^{1/2}$ for the existence of a baby-skyrmion in a GaAs quantum well at $\nu = 1$. This energy is far below the value $-0.17 \times 4\pi\rho_s$ previously calculated by using a variational approximation for the microscopic wave function of the charged spin texture.[3,4]

Our scheme can be directly applied to a QHF corresponding to a filling factor $\nu = 1/3$, just by changing the energy scale given by the spin stiffness that now is 27 times smaller than in the case of filling factor 1.[10] This implies a critical g factor $g_{cr} = 0.29/\sqrt{B}$ T$^{1/2}$ for the existence of a baby-skyrmion in a GaAs quantum well at $\nu = 1/3$. For a field of a few Teslas, this value is below the g-factor of GaAs. This would explain why skyrmions at $\nu = 1/3$ are observed only if high pressure is applied reducing the g-factor,[11] while they do not appear in the normal case.[11,12]

C. Scattering of a single exciton by a defect of charge $\pm Z$

Our model is easily generalized to the case in which $Z > 1$ flux quanta are added or removed from the system in order to create a defect with higher real and topological charge. In this case the metric is given by

$$\mu_1^{(Z)}(z) = e^{\pm iZ\theta}\sqrt{1 - 2\pi l_B^2 \sum_{m=0}^{Z-1} |\varphi_m(z)|^2}. \tag{17}$$

Z quasiholes or quasi electrons are occupying $\varphi_m(z)$ states (with $m = 0, ..., Z-1$) due to the added or removed flux quanta. Now the metric vanishes as r^{2Z} near the origin, so that an exciton wave function diverging at the origin as r^{-Z} has finite norm. The potential $V(z)$ has the same behavior described for $Z = 1$. In particular, its minimum is equal to the energy of the skyrmion with topological charge Z predicted by the NLσM, i.e., $-Z4\pi\rho_s$.

Once again the product of functions $\mu_1^{(Z)}(z)\phi_1^Z(z)$ in the lowest energy solution of Eq. (14) has no angular dependence, so that the exciton wave function behaves as $e^{\mp iZ\theta}$. In other words, we find that when a spin-wave becomes bound to a defect of physical charge Z, it raises a spin texture $O_{\phi z}^{\dagger}|^{Z-1}F\rangle$ with winding number Z.

As an example, we have computed the binding energy of one exciton in a charged defect with $Z = 2$. We get an energy of $-1.13 \times 4\pi\rho_s$ which is much smaller than the result obtained from a Hartree-Fock calculation.[13] This energy is important to decide if a skyrmion with charge $Z = 2$ is cheaper or not than two skyrmions of charge $Z = 1$ separated a finite distance.

III. SEVERAL SPIN-WAVES BOUND TO A CHARGED DEFECT

In the previous section we have focused our attention on spin textures made up with only one electron flipping its spin. Since the Zeeman contribution to the energy is separable from the contribution due to the interaction between electrons, one can add it *a posteriori* so that, hitherto, we have not included it. The Zeeman energy is proportional to the number of electrons that have flipped their spin to raise the texture. As rotations in spin space around the direction of the magnetic field are symmetry operations, this is a well defined quantum number for the textures. In our language, in which we count each spin flip as an exciton, the Zeeman coupling constant just plays the role of a chemical potential. By tuning the g-factor, we may change the number of excitons actually present in the lowest lying texture. In this section we are going to study those textures made up of more than one bound exciton. Once again, the Zeeman contribution is added at the end of the process.

We will approach this problem within a mean field theory for the excitons. Our task then reduces to generalize the scattering formalism we have already developed, *i.e.* Eq. (14), in order to describe the dynamics of one single exciton in the presence of both the unbalanced flux quantum and a background of the remaining excitons. We want to stress that the singled out exciton is identical to those in the background, and so we will be led to solve this problem self-consistently. It is also worthy to comment that if we apply the same ideas to the spin-waves in the uniform QHF, we obtain that they do not interact because they are completely delocalized. This is not the case, nevertheless, for the excitons bound to the quasiparticle. Before going into formal matters, let us analyze what we should expect to be the effect of a background of excitons in the dynamics of another one coming into the region of the defect.

When an exciton gets bound, spin up hole states around the charge defect begin to be filled up. A new exciton moving in this texture will find a density of occupied (unaccessible) hole states larger than it would find if there were only a bare defect. Therefore, we expect a background of already bound excitons to change the effective metric felt by a new coming one in order to account for the new unaccessible states. Moreover, as some exchange energy has already been gained by trapping the background excitons, we also should expect a less attractive effective potential and a weaker energy needed to keep a single exciton tied in the spin texture. In other words, we expect the background of bound excitons to screen the bare defect. As more excitons get bound, the effective screening spreads over a larger region, and so does the area where the excitons are constrained to stay. Eventually, when the flux quantum is almost completely screened, the energy required to tie or drop a single

exciton tends to vanish, and the system develops order in the magnetization. The resulting textures are skyrmions of quasi infinite size, as the ones described by a NLσM.

Let us formally derive now the one particle effective Hamiltonian for an exciton in the presence of both a flux quantum and a set of $K-1$ previously bound excitons. The first thing to do is to deduce the change in the metric due to the binding of more and more excitons. At the K-th step, the mean field state is built up as $(O_\phi^\dagger)^K|0F\rangle = O_\phi^\dagger\left[(O_\phi^\dagger)^{(K-1)}|0F\rangle\right]$. Note that all the K excitons are in the same state $\phi_K(z)$. The only reason for writing the K-th separately is to make explicit our aim of studying the dynamic of a single one of them in the presence of the remaining ones.

Following our own steps in the previous section, let us evaluate the normalization condition for the exciton wave function $\phi_K(z)$

$$1 = \frac{1}{K}\frac{\langle^0F|(O_\phi)^{(K-1)}O_\phi O_\phi^\dagger(O_\phi^\dagger)^{(K-1)}|^0F\rangle}{\langle^0F|(O_\phi)^{(K-1)}(O_\phi^\dagger)^{(K-1)}|^0F\rangle} \tag{18}$$

where the factor $1/K$ appears due to the different possibilities of singularizing one exciton among the K electrons and holes which are present. If the operators $\Psi_\downarrow^\dagger(\mathbf{r})\Psi_\uparrow(\mathbf{r})$ and $\Psi_\uparrow^\dagger(\mathbf{r}')\Psi_\downarrow(\mathbf{r}')$ satisfied perfectly bosonic commutation relations, Eq.(18) would imply again the condition for normalization of a unique exciton

$$\int dz|\phi_K(z)|^2|\mu_1(z)|^2 = 1 \tag{19}$$

with $\mu_1(z)$ given by Eq. (11). In fact this is the case for few spin waves excited on the uniform QHF, and they behave as independent bosons. However, in the presence of the defect of charge it is not so anymore. What we get instead from Eq.(18) is

$$\int dz|\phi_K(z)|^2|\mu_K(z)|^2 = 1 \tag{20}$$

where the new renormalized metric is

$$|\mu_K(z)|^2 = |\mu_1(z)|^2 - |\Delta\mu_K(z)|^2 \tag{21}$$

with the variation of the metric taking the form

$$|\Delta\mu_K(z)|^2 = (K-1)\sum_{m>0}\alpha_m^{(K)}|\varphi_m(z)|^2. \tag{22}$$

In this expression, $\varphi_m(z)$ are the single-electron states given by Eq. (8) and $\alpha_m^{(K)}$ satisfy

$$\sum_{m>0}\alpha_m^{(K)} = 1 \tag{23}$$

for any K. For each step K, $\alpha_m^{(K)}$ is a function (given in the Appendix up to $K=4$) of the wave function of the exciton through the integrals $\int dz|\phi_K(z)|^2|\varphi_m(z)|^2$. Apart from the more or less complicated expression of these coefficients, Eq. (21) simply states that the initial metric $\mu_1(z)$ is corrected by the non zero occupation of states $\varphi_m(z)$ with $m>0$ due to the $K-1$ previously bound excitons. The condition (23) guarantees that the number

10

$\int dz |\Delta\mu_K(z)|^2$ of new unaccessible hole states equals the number $K-1$ of previously bound excitons. Therefore, the coefficients $\alpha_m^{(K)}$ play the role of effective occupations of states $\varphi_m(z)$ in the texture.

It must be pointed out that, as the coefficients $\alpha_m^{(K)}$ involve a dependence on the excitonic wave function $\phi_K(z)$, $|\mu_K(z)|^2$ is effectively a new renormalized metric only for the very same wave function with which we have built it. Nevertheless, at first order, we may accept that it is also valid for all those other wave functions that are close in energy to the former one. If the constant function, $\phi(z) \equiv constant$, could be included among these, we would just need to repeat the symmetry reasoning of the previous section to obtain a new effective exchange potential, and so the generalization of Eq. (14) we are looking for:

$$H_K \mu_K \phi_K = 4\pi\rho_s \left[-\nabla^2 + \frac{\nabla^2 \mu_K(z)}{\mu_K(z)} \right] \mu_K \phi_K = \varepsilon_K \mu_K \phi_K. \qquad (24)$$

The reason why we can actually go this step lies on the weak binding energy of a single exciton. It was only one third of the depth of the potential for the first bound exciton, and will be even smaller as more excitons are tied.

For each number K of excitons, this equation must be solved self-consistently due to the dependence of μ_K on ϕ_K. Such a task must be performed numerically. In this way we have obtained both ϕ_K and the energy ε_K up to $K = 4$. As expected, we obtain a set of metrics, $|\mu_1(\phi_1)|^2$, ... , $|\mu_4(\phi_4)|^2$ which spread their inhomogeneous core for increasing K. This implies potentials $V_1(\phi_1)$, ... , $V_4(\phi_4)$ which are successively less attractive and more extended as shown in Fig. 1. As a direct consequence, the states $\mu_1\phi_1$, ... , $\mu_4\phi_4$ are progressively less localized.

Once we have the self-consistent solution for an exciton in the presence of both the flux quantum and a background of $K-1$ excitons, the mean field wave function describing K excitons bound to the defect is just

$$\Phi(z_1, ..., z_K) = \phi_K(z_1)...\phi_K(z_K) \qquad (25)$$

The energy E_K of this state must be computed with a little care. If we just considered $E_K = K\varepsilon_K$, the exciton-exciton repulsion would be overestimated. Instead, we must pill up the excitons one by one in the final state, summing up all the energies required in the operation. We must then compute the expected values of effective Hamiltonians built up with $0, 1, 2, ..., K-1$ excitons which are identical to the ones obtained self-consistently in the presence of a background of $K-1$ excitons. Labeling the corresponding effective Hamiltonians as $\mathcal{H}_1(\phi_K)$, ... , $\mathcal{H}_K(\phi_K)$, the energy becomes

$$E_K = \sum_{j=1}^{K} \langle \phi_K | \mathcal{H}_j(\phi_K) | \phi_K \rangle. \qquad (26)$$

Table I gives our results for each term of Eq. (26) as well as for the energies E_K. Since $\mathcal{H}_K(\phi_K) \equiv H_K$, the diagonal of the left side of the table directly gives the eigenvalues ε_K. Our results verify all the expected behaviors about progressively less bound excitons and decreasing total energy as K increases. Repulsion between different excitons can also be drawn from Table I.

11

As discussed above, by repeating the procedure indefinitely, for $K = \infty$, one should obtain the classical skyrmion of the NLσM. This is obviously impossible from the practical point of view but we can use a Wynn's algorithm[14] to extrapolate our results in Table I and estimate their asymptotic limit for the texture with $K \to \infty$. Such extrapolation (also included in table I) gives $E_\infty = -1.03 \times 4\pi\rho_s$, while the NL$\sigma$M gives for skyrmions an energy $-4\pi\rho_s$ with respect to $|^0F\rangle$. Taking into account the intrinsic uncertainties of the extrapolation procedure, this result can be considered as satisfactory enough to conclude that our scattering procedure recovers the NLσM skyrmion when infinite excitons are bound to the defect.

The energies displayed in Table I are significantly lower than the ones obtained from Hartree-Fock calculations[2] or variational procedures.[4] Therefore, we would obtain larger textures than those predicted by previous calculations. By extrapolating the energies given in Table I, we get, for the g-factor of GaAs, a skyrmion with $K = 7$ for $B = 4$T and $K = 5$ for $B = 20$T. This is too large compared with the size experimentally estimated[15,16] because we are not taking into account both the finite width of the wells and the effect of higher Landau levels. Both effects tend to reduce the strength of the interactions with respect to the Zeeman term. This implies a smaller size of the skyrmion what means a better agreement with experiments.

We still owe an explanation of why the angular momentum of the excitons in their ground state equals the winding number of the spin textures resulting from a condensation of many of those excitons. Hitherto, we have built many-body wave functions

$$\left[O_\phi^\dagger\right]^K |^0F\rangle = \left[\int dz\, f(|z|)e^{-i\theta}\Psi_\downarrow^\dagger(z)\Psi_\uparrow(z)\right]^K |^0F\rangle \tag{27}$$

containing a well defined number K of excitons (i.e. of spin flips) to preserve the symmetry under rotations in spin space around the direction of the magnetic field. The expected value of the in-plane (perpendicular to the magnetic field) magnetization in these states is then zero everywhere. Nevertheless, for vanishing Zeeman coupling, when many electrons find it cheaper to flip their spin, we have just seen that the energy required for the texture to tie or drop a single exciton ($\lim_{K\to\infty} E_K - E_{K\pm1}$) vanishes. Then, a coherent superposition of states containing whatever number of excitons is allowed, and the system develops order in the magnetization. This coherent state is described by the BCS-like wave function[2],

$$\prod_{m\geq0}(c_{m+1,\uparrow}^+ + u_m e^{i\varphi}c_{m,\downarrow}^+) |0\rangle. \tag{28}$$

The projections of this wave function onto the subspaces of states with well defined number K of spin flips (excitons) are just our mean field states[5]

$$\left[O_\phi^\dagger\right]^K |^0F\rangle = C \int_0^{2\pi} d\varphi\, e^{-iK\varphi} \prod_{m\geq0}(c_{m+1,\uparrow}^+ + u_m e^{i\varphi}c_{m,\downarrow}^+) |0\rangle. \tag{29}$$

$c_{m,\sigma}^+$ is the representation of the electron field operator $\Psi_\sigma^\dagger(z)$ in the states with well defined angular momentum, φ_m, given by Eq. (8). $|0\rangle$ is the vacuum state, and C is a constant. The degrees of freedom $f(|z|)$ in (27) and u_m in (28) are simply related to each other by the change of representation. Note that excitons with unit angular momentum, only mix electron states differing also in one unit of angular momentum.

The BCS-like wave function (28) describes, for slowly decaying $f(|z|)$, a spin texture with topological charge 1. The decay $f(|z|) = 1/|z|$, that we have found in the previous section to minimize the energy for vanishing Zeeman coupling, is an antiskyrmion. In this latest case, the BCS-like wave function (28) reads simply in first quantization[5]

$$\prod_{i=1}^{N} \begin{pmatrix} z_i \\ \xi e^{i\varphi} \end{pmatrix} |F\rangle \tag{30}$$

that is the usual representation for antiskyrmions. The parenthesis is the spinor of the i-th electron and z_i its position in the plane. The parameter ξ gives the size of the antiskyrmion.

IV. CONCLUSIONS

We have presented a theoretical framework to describe skyrmions in QHF as a condensate of spin-waves bound to a bare charged defect. The scheme can be easily applied to QHF corresponding to filling factors $1/(2p+1)$, with p integer.

The low lying spectrum of neutral excitations of a QHF (spin-waves) has been mapped to that of free moving spin excitons of vanishing electrical dipolar moment. We have then studied their scattering by a bare quasiparticle, and we have found that it may bind them. So the charged excitation carries a spin texture. The bound excitons interact with the bare defect of charge as well as with each other. The effects of the former are:

i) Pauli principle does not allow the excitons to be at the same position as the defect. We describe this non-uniform space where the exciton moves by an inhomogeneous metric in the Hilbert space of its wave functions.

ii) The defect of charge leaves a quantum of magnetic flux unbalanced. This introduces an Aharonov-Bohm phase that gives a unit of angular momentum to the excitons in their ground state. The resulting texture has winding number equal to 1.

iii) The localization of the spin of the bare quasiparticle implies a waste of exchange energy. The exchange energy that may be gained by a smoother alignment of spins is felt as an attractive potential where the excitons become bound. This potential preserves the symmetry under rotations in spin space in the inhomogeneous space described in i).

In this work we have deduced explicit expressions for both the inhomogeneous metric and the potential. Our framework is supported by comparison with results obtained by numerical calculations.

To take into account the interaction among bound excitons we have used a mean field approximation for them. We find that it works quite well not only for textures containing many excitons, but also for those others containing a few of them.

As a final comment, we must stress that all the states we obtain have a well defined third component of the total spin. Therefore, it is not possible to associate to those states a local vector field having the characteristics of a spin texture. In order to get it, it is necessary to allow for linear combinations of states with different values of the third component of the spin.[5] This is possible for vanishing Zeeman energies that favors states containing many excitons, because all of them are almost degenerate in energy. The resulting texture is a skyrmion.

V. ACKNOWLEDGEMENTS

APPENDIX A: COEFFICIENTS $\alpha_M^{(K)}$ FOR THE VARIATION OF THE METRIC

The coefficients $\alpha_m^{(K)}$ giving the variation of the metric in Eq.(22) are rational functions of the variables

$$\beta_{m'}^{(K)} = \int dz |\phi_K(z)|^2 |\varphi_{m'}(z)|^2 \tag{A1}$$

with m' going from 1 to infinity. Up to 4 excitons, these rational functions are:

$$\alpha_m^{(1)} = 0 \tag{A2}$$

$$\alpha_m^{(2)} = \frac{\beta_m^{(2)}}{\sum_{m'>0} \beta_{m'}^{(2)}} \tag{A3}$$

$$\alpha_m^{(3)} = \frac{\beta_m^{(3)} \sum_{m'} \beta_{m'}^{(3)} - [\beta_m^{(3)}]^2}{\left(\sum_{m'} \beta_{m'}^{(3)}\right)^2 - \sum_{m'} [\beta_{m'}^{(3)}]^2} \tag{A4}$$

$$\alpha_m^{(4)} = \frac{\beta_m^{(4)} \left(\sum_{m'} \beta_{m'}^{(4)}\right)^2 - 2[\beta_m^{(4)}]^2 \sum_{m'} \beta_{m'}^{(4)} + 2 \sum_{m'} [\beta_{m'}^{(4)}]^3 - \beta_m^{(4)} \sum_{m'} [\beta_{m'}^{(4)}]^2}{\left(\sum_{m'} \beta_{m'}^{(4)}\right)^3 - 3 \left(\sum_{m'} \beta_{m'}^{(4)}\right) \sum_{m'} [\beta_{m'}^{(4)}]^2 + 2 \sum_{m'} [\beta_{m'}^{(4)}]^3} \tag{A5}$$

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FIGURES

FIG. 1.   Effective potential $\nabla^2 \mu_K(z)/\mu_K(z)$ for $K = 1, .., 4$ as a function of the distance $|z|$ (in units of the magnetic length) to the charged defect. The eigenvalues ε_K are also shown.

TABLES

TABLE I. Binding energies $\langle\phi_K|\mathcal{H}_j(\phi_K)|\phi_K\rangle$ of the j-th exciton in the K-th step (see text) and total energy E_K of the texture with K excitons bound to the charged defect in units of $4\pi\rho_s$. The total energy $E_{K\to\infty}$ obtained by an extrapolation (see text) is also included. The diagonal of the left side gives the eigenvalues ε_K.

	$\langle\phi_K|\mathcal{H}_1(\phi_K)|\phi_K\rangle$	$\langle\phi_K|\mathcal{H}_2(\phi_K)|\phi_K\rangle$	$\langle\phi_K|\mathcal{H}_3(\phi_K)|\phi_K\rangle$	$\langle\phi_K|\mathcal{H}_4(\phi_K)|\phi_K\rangle$	E_K
K=1	-0.311				-0.311
K=2	-0.304	-0.191			-0.495
K=3	-0.295	-0.201	-0.138		-0.634
K=4	-0.287	-0.205	-0.149	-0.109	-0.750
K→ ∞					-1.030